



High-frequency Rayleigh waves in materials with micro-structure and couple-stress effects

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Abstract

A well-known deficiency of the classical theory of elasticity is that it does not predict dispersive Rayleigh-wave motions at any frequency. This contradicts experimental data and predictions of the discrete particle theory (atomic-lattice approach) for high frequencies. The present work is intended to explore whether the elastic couple-stress theory with micro-structure can overcome the deficiency of the classical theory. Our analysis shows indeed that Rayleigh waves propagating along the surface of a half-space are dispersive at high frequencies, a result that can be useful in applications of high-frequency surface waves where the wavelength is often of the micron order. Provided that certain relations hold between the various micro-structure parameters entering the theory employed here, the dispersion curves of these waves have the same form as that given by previous analyses based on the atomic-lattice theory. In this way, the present analysis gives means to obtain estimates for micro-structure parameters of the couple-stress theory. Besides the Rayleigh-wave results reported here, basic theoretical results for the kinetic energy and momentum balance laws in micro-structured media with couple-stress effects are derived and presented.

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1. Introduction

The present work is concerned with Rayleigh-type surface wave propagation in a material with micro-structure. To explain *dispersion phenomena* at high frequencies (small wavelengths) occurring in practical situations (see e.g. Gazis et al., 1960; White, 1970; Farnell, 1978) and therefore to circumvent the deficiency of the conventional elasticity theory (which does not predict dispersion of Rayleigh waves at *any* frequency), the problem is attacked with the couple-stress theory of elasticity with micro-structure. Indeed, it is expected that material micro-structure will be an important factor in the propagation of high-frequency surface waves often encountered in electronic-device applications, since the frequencies are on the order of GHz (or even greater) and therefore wavelengths on the micron order may appear. The theory employed

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here falls into the category of generalized continuum theories and is a particular case of the general approaches of Mindlin (1964) and Green and Rivlin (1964). In these theories, the basic idea consists of endowing each point of a continuum with an *internal* displacement field, which is expanded as a power series in internal coordinate variables. The lowest-order theory is obtained by retaining only the first (linear) term. As is well-known, ideas underlying the couple-stress linear theory of elasticity were advanced first by Voigt (1887) and the brothers Cosserat and Cosserat (1909), but the subject was generalized (e.g. including inertia effects) and reached maturity only with the aforementioned works of Mindlin and Green/Rivlin. In addition, Kröner (1963) gave physical aspects pertinent to crystal lattices and a *non-local* interpretation of the theory.

It is noted that earlier application of the couple-stress elasticity theory, mainly on quasi-static problems of stress concentration, met with success providing solutions more adequate physically than classical-elasticity solutions (see e.g. Mindlin and Tiersten, 1962; Weitsman, 1965, 1967; Day and Weitsman, 1966; Bogy and Sternberg, 1967a,b; Lakes, 1982). Extensive work employing couple-stress theories, as well as related strain-gradient theories, on elasticity and plasticity problems is also continued in recent years (see e.g. Papamichos et al., 1990; Anderson and Lakes, 1994; Fleck et al., 1994; Vardoulakis and Sulem, 1995; Lakes, 1995; Wei and Hutchinson, 1997; Huang et al., 1997; Chen et al., 1998; Georgiadis, 2000; Lubarda and Markenscoff, 2000; Bardet and Vardoulakis, 2001; Amanatidou and Aravas, 2001). In particular, Huang et al. (1997) provided solutions to interesting crack problems, Lubarda and Markenscoff (2000) derived general conservation laws for the couple-stress elasticity theory, and Bardet and Vardoulakis (2001) discussed the importance of couple-stresses in granular media.

In the present work, although complexity of the theory has been kept at a minimum (by retaining a restricted number of material parameters), a *micro-inertia* term was included (in a rigorous manner) because previous experience with related wave-propagation problems considered through the dipolar gradient theory (Georgiadis et al., 2000) shows that this term is indeed important at high frequencies and including it in the formulation of the problem gives, in fact, dispersion curves that mostly resemble with the ones obtained by atomic-lattice considerations. It is also noticed that no such couple-stress theory with micro-structure was proposed up to now to deal with Rayleigh-wave motions. Recently, however, the dipolar gradient theory *without* couple-stress effects was employed to study the same problem (Georgiadis et al., 2002) and some other wave-propagation problems (Vardoulakis and Georgiadis, 1997; Georgiadis and Vardoulakis, 1998; Georgiadis et al., 2000). Finally, we emphasize that contrary to quasi-static couple-stress approaches, which do not include *explicitly* the size of the material unit cell (micro-medium) in the formulation of the problem, the present approach fully takes into account this intrinsic material length by appropriately considering a micro-inertia term in the balance of angular momentum. Moreover, the present analysis provides means to estimate the relation between the characteristic material length $2h$ and the so-called couple-stress modulus η (which is the coefficient of the rotation gradient, in the strain-energy density expression, and is an additional—to the standard Lamé constants—material parameter in the couple-stress theory). This, in general, can be obtained through comparisons of dispersion curves that can be obtained in the spirit of the present analysis with dispersion curves given either by experiments or atomic-lattice calculations.

Another important notice pertains to the relevance of the couple-stress theory in modeling Rayleigh-wave motions. Indeed, the physical mechanism of the Rayleigh-wave motions (as this mechanism is revealed, e.g. in the experimental work of Dally and Thau, 1967) suggests that there is a much stronger shear contribution than a dilatational one, and therefore one may expect that considering pronounced shear effects in the material response (as is the effect of the gradient of *rotation* and the associated effect of *non-collinear* dipolar internal forces resulting in couples that are included in the present formulation) most properly simulates the phenomenon of Rayleigh waves.

In our analysis, displacement potentials of the Lamé type and two-sided Laplace transforms in the complex domain are employed. Results are then derived through numerically solving the dispersion

equation by techniques of root bracketing and root finding with bisection (Brent's method—see e.g. Press et al., 1986). Here also, besides the main analysis on Rayleigh waves, a development of basic theoretical results concerning the kinetic energy and momentum balance laws in materials with micro-structure and couple-stress effects is provided.

Before presenting our own results, we would like to discuss briefly two related studies published before on the subject of dispersive Rayleigh waves. The first study (Suhubi and Eringen, 1964) develops and applies the theory of micro-polar elasticity. This theory is more general than the one employed here but, also, much more complicated since it involves 18 material constants (in its isotropic linear version) and the present theory involves only four material constants (see Eq. (31) below). The work by Suhubi and Eringen (1964) on Rayleigh waves, however, does not employ explicitly the size of the unit cell (intrinsic material length) nor present any dispersion curve. Also, they reached the final dispersion equation through some approximation. Finally, in both the general Mindlin–Green–Rivlin theory and the simpler Eringen–Suhubi theory, the large number of material constants prevents their applicability to practical problems and poses difficulties in physically interpreting and measuring all these constants.

The second study (Ottosen et al., 2000) employs the linear couple-stress theory without micro-structure. The absence of considerations regarding micro-structure (i.e. ignoring the micro-inertia term) in the latter work is the main difference with the present work. Of course, ignoring the micro-structure in this way deprives one to extract any relation between the couple-stress modulus η and the intrinsic material length h . Besides that, ignoring the micro-structure is a simplification in any event. As explained below (see relative comments immediately after Eq. (25)), the difficulty with the non-objectivity of the stress tensor (the non-objectivity is due to the micro-inertia term and the body-couple field) can be circumvented even in the general transient case and be completely *eliminated* in the specific time-harmonic case considered here to analyze the Rayleigh waves. Thus, use of the complete theory has no danger of ill-posedness in the problem considered here. In addition, the results given in Ottosen et al. (2000) show that the Rayleigh-wave velocity *exceeds* the shear-wave velocity for a rather wide range of wavenumber values, as the frequency increases. As these authors admit, this is a physically questionable result showing that the couple-stress theory without micro-structure is not so successful in analyzing Rayleigh waves. On the contrary, the present analysis is able to control the form of dispersion curves by properly adjusting the values of the couple-stress modulus η in relation with the intrinsic material length h . Finally, the work by Ottosen et al. (2000) follows a different procedure than ours in deriving the dispersion equation. For instance, they do not use the analysis of Muki and Sternberg (1965), who elucidated the issue of boundary conditions in plane-strain problems of couple-stress elasticity. As a consequence, Ottosen et al. (2000) work with five boundary conditions instead of the three boundary conditions pertinent to the plane-strain case. This complicates their solution procedure. Also, they do not present dispersion curves. Instead, it is shown here that deriving dispersion curves is important because these may serve for comparisons with experimental results and/or atomic-lattice results.

2. Basic concepts and equations

In this section we briefly give the basic ideas and equations of the couple-stress theory of elasticity with the effects of micro-structure and inertia/micro-inertia. The theory employed here is a particular case of Form III in the general Mindlin's (1964) approach. Nevertheless, we chose to present an alternative approach to Mindlin's variational approach. Indeed, our derivation of basic results relies on the momentum balance laws, which—in our opinion—provide more physical insight. It should also be mentioned that versions of the couple-stress theory were introduced by, among others, Mindlin and Tiersten (1962), Koiter (1964), Weitsman (1965), Muki and Sternberg (1965), and Mindlin and Eshel (1968). However, the latter formulations do *not* include inertia and micro-inertia effects since they are of quasi-static character. As a

consequence, the size of the material cell is not explicitly included in the governing equations of these approaches and of other recent approaches of quasi-static character. In these analyses, rather, a characteristic length appears in the governing equations only through the ratio (η/μ) , where η is the couple-stress modulus (having dimensions of [force]) and μ is the standard shear modulus of the material. Indeed, the ratio (η/μ) has dimensions of [length]². As will become apparent below, it is only the dynamic analysis that explicitly accounts for the size of the material unit cell (granule). Of course, the ratio (η/μ) also appears within the dynamic analysis, which therefore may allow for an interrelation of the two characteristic lengths mentioned above.

The point of departure is considering a generalized continuum with material particles (macro-volumes), behaving like *deformable* bodies. This behavior can easily be realized if such a macro-volume is viewed as a collection of sub-particles (see Fig. 1). It is further assumed that *internal* forces (called dipolar forces by Green and Rivlin (1964) and double forces by Mindlin (1964)) are developed between the sub-particles (see Fig. 2). Although each pair of the dipolar forces has a zero resultant force, it gives generally a *non-zero* moment and therefore gives rise to stresses on a surface called couple-stresses. This means that a surface element may transmit, besides the usual *force* vector, a *couple* vector as well. In this way, the Euler–Cauchy stress principle of classical (monopolar) continuum mechanics is augmented by considering additional couple-tractions. One can interpret physically the couple-stresses as created by frictional couples resisting the relative rotation of the grains (sub-particles). We note that examples of force systems of the dipolar collinear or non-collinear type are given by Fung (1965, p. 304). Also, it is emphasized that although the dipolar forces are self-equilibrating they produce generally non-vanishing stresses, the dipolar stresses. Here, we consider only couple-stress effects and, accordingly, we have assumed that *only* the anti-symmetric part of the dipolar forces contributes to the stress field. Compatible with this assumption is the choice of a form of the strain-energy density (cf. Eq. (31) below) that depends upon the strain and the gradient of rotation, but does not depend upon the gradient of strain.

2.1. Kinetic energy and momentum balance laws

A typical material particle occupies a volume V (material volume). Each sub-particle (micro-medium) has a mass m_P ($P = 1, 2, \dots$), whereas the mass center of a typical sub-particle has coordinates \mathbf{x}_P or x_{Pi} ($i = 1, 2, 3$) with respect to a Cartesian coordinate system x_j ($j = 1, 2, 3$). The volume of the material particle (macro-medium) has a total mass $m = \sum m_P$, where summation is understood over P from 1 to the number of sub-particles comprising the macro-medium. The mass center of the material particle is given as

$$\mathbf{c} = \frac{\sum m_P \mathbf{x}_P}{m} = \frac{\int_V \mathbf{x} dm}{m}. \quad (1)$$

The quantities $\sum m_P x_{Pi}$ and $\sum m_P x_{Pi} x_{Pj}$ are, respectively, the first and second moments of mass. We also define the relative positions of sub-particles

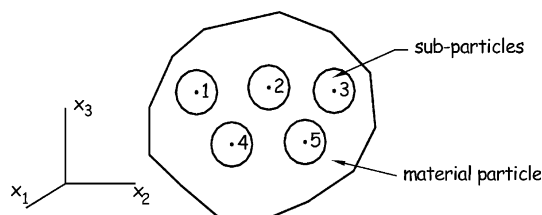


Fig. 1. A continuum with micro-structure. A material particle is composed by sub-particles.

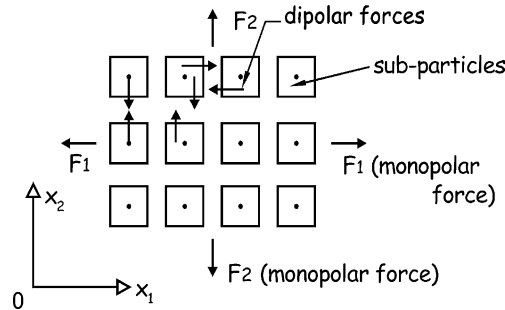


Fig. 2. Monopolar (external) and dipolar (internal) forces acting on an ensemble of sub-particles in a material with micro-structure. The anti-symmetric part of the dipolar forces gives rise to couple-stresses.

$$\bar{\mathbf{x}}_P = \mathbf{x}_P - \mathbf{c}. \quad (2)$$

Then, in view of (2), the second moment of mass takes the form

$$\sum m_P \mathbf{x}_{Pi} \mathbf{x}_{Pj} = m c_i c_j + \sum m_P \bar{\mathbf{x}}_{Pi} \bar{\mathbf{x}}_{Pj}. \quad (3)$$

From Eq. (3) next, differentiating w.r.t. time and by taking also into account that $m\dot{\mathbf{c}} = \sum m_P \mathbf{v}_P$ and $\sum m_P \bar{\mathbf{v}}_P = 0$ (the latter equality actually means that the ‘internal velocities’ of the sub-particles do *not* contribute to the linear momentum—see below for the proof), one obtains

$$\sum m_P \mathbf{x}_{Pi} v_{Pj} = m c_i \dot{c}_j + \sum m_P \bar{\mathbf{x}}_{Pi} \bar{v}_{Pj}, \quad (4)$$

where $(\dot{}) \equiv D()/Dt$ is the material time-derivative, $v_{Pj} \equiv \dot{x}_{Pj}$ and $\bar{v}_{Pj} \equiv \dot{\bar{x}}_{Pj}$.

The simplest possible mode of *internal* motion is now assumed, i.e. the *linear* approximation

$$\bar{v}_{Pj} = v_{jk} \bar{x}_{Pk} \quad (k = 1, 2, 3), \quad (5)$$

where v_{jk} can be called *dipolar* velocities. These can be considered the counterparts of the velocity gradients $\partial_k v_j$ (with $\partial_i() \equiv \partial()/\partial x_i$) describing the rates of deformation of the ‘network’ of lines connecting the mass centers of the sub-particles. In the same manner, one can introduce higher-order multipolar velocities by writing $\bar{v}_{Pj} = v_{jk} \bar{x}_{Pj} + v_{jkl} \bar{x}_{Pk} \bar{x}_{Pl} + \dots$, with $(l = 1, 2, 3)$, if the internal motions were to be described in greater detail. However, in doing this, a much more complicated theory results than the dipolar theory. That theory will also involve tripolar, quadrupolar and so on, velocities and forces. It is obvious that the increased complexity of such a theory does not hold much hope for treating practical problems.

Further, from Eqs. (4) and (5), one obtains

$$\sum m_P \mathbf{x}_{Pi} v_{Pj} = m c_i \dot{c}_j + I_{ik} v_{jk}, \quad (6)$$

where

$$I_{ik} = \sum m_P \bar{\mathbf{x}}_{Pi} \bar{\mathbf{x}}_{Pk}. \quad (7)$$

Eq. (7) shows that the quantities I_{ik} depend upon the ‘arrangement’ of the masses of sub-particles within the volume of the material particle.

In view of the above, the *kinetic energy* of the material particle with volume V is decomposed now as

$$T^{(V)} \equiv \frac{1}{2} \sum m_P v_{Pi} v_{Pi} = \frac{1}{2} m \dot{c}_i \dot{c}_i + \frac{1}{2} I_{jk} v_{ij} v_{ik}, \quad (8)$$

or, equivalently (due to (3), (5) and (7)), as

$$T^{(V)} = \frac{1}{2} m \dot{\mathbf{c}} \cdot \dot{\mathbf{c}} + \frac{1}{2} \int_V \bar{\mathbf{v}} \cdot \bar{\mathbf{v}} \, dm. \quad (9)$$

Finally, we consider the case of a homogeneous continuum with couple-stresses. The assumption of homogeneity is a rather standard one in recent studies applying the Mindlin–Green–Rivlin theory to practical problems and implies that the relative deformation (i.e. the difference between the macro-displacement gradient and the micro-deformation—cf. Mindlin, 1964) is zero and also that the micro-density does not differ from the macro-density. The continuum is composed wholly of unit cells having the form of cubes with edges of size $2h$ (h is therefore a characteristic dimension) and has mass density ρ . Then, compatible with couple-stress effects is taking the material particles as ‘rigid’ so that the dipolar velocity v_{ij} above is replaced by an intrinsic *angular* velocity. In view of the above, the geometrically *linear* theory gives the following expression for the kinetic-energy density (kinetic energy per unit macro-volume)

$$T = \frac{1}{2} \rho \dot{u}_i \dot{u}_i + \frac{\rho h^2}{3} \dot{\omega}_i \dot{\omega}_i, \quad (10)$$

where u_i are the components of the displacement of the material particle, $\omega_i \equiv (1/2)e_{ijk}\partial_j u_k$ is the rotation vector with e_{ijk} being the permutation symbol, and indicial notation is understood.

Next, we consider the *linear* and *angular momentum* of the material particle. First, the linear momentum of a macro-volume V (ensemble of a number of micro-media) of the material is written as

$$\mathbf{L} \equiv \int_V \mathbf{v} \, dm = \int_V \dot{\mathbf{x}} \, dm, \quad (11)$$

which by virtue of (1) becomes

$$\mathbf{L} = m \dot{\mathbf{c}}. \quad (12)$$

As promised before, we now prove that $\int_V \bar{\mathbf{x}} \, dm = 0$ and, therefore, that the ‘internal velocities’ do *not* contribute to the linear momentum, i.e. $\int_V \bar{\mathbf{v}} \, dm = 0$. The proof is simple and runs as follows: $\mathbf{c}m - \mathbf{c} \int_V dm = 0 \iff \int_V \mathbf{x} \, dm - \int_V \mathbf{c} \, dm = 0 \iff \int_V (\mathbf{x} - \mathbf{c}) \, dm = 0 \iff \int_V \bar{\mathbf{x}} \, dm = 0$.

In addition, the angular momentum about the coordinate origin is written as

$$\mathbf{H}^{[0]} \equiv \int_V (\mathbf{x} \times \mathbf{v}) \, dm = \int_V [(\mathbf{c} + \bar{\mathbf{x}}) \times \mathbf{v}] \, dm = \mathbf{c} \times \int_V \mathbf{v} \, dm + \int_V (\bar{\mathbf{x}} \times \mathbf{v}) \, dm \equiv (\mathbf{c} \times \mathbf{L}) + \mathbf{H}^{[c]}, \quad (13)$$

where $\mathbf{H}^{[c]}$ denotes the angular momentum about the mass center of the material particle.

As for the forces acting on the material particle and the sub-particles, along with the resultant force $\mathbf{F} = \sum \mathbf{F}_P$, the moments $F_{ij} = \sum F_{Pj} \bar{x}_{Pi}$, $F_{ijk} = \sum F_{Pj} \bar{x}_{Pi} \bar{x}_{Pk}$, etc. can be considered so that, by virtue of (5), the mechanical power is written as $\mathcal{M} = \sum F_{Pi} v_{Pi} = F_i \dot{c}_i + F_{ij} v_{ij} + F_{ijk} v_{ijk} + \dots$. In this way, the so-called multi-polar forces (F_{ij}, F_{ijk}, \dots) are introduced as coefficients representing *higher-order* forces in the expansion of the mechanical power \mathcal{M} .

Finally, in view of the foregoing discussion, the momentum *balance laws* for a control volume (consisting of a number of macro-media) of the body can easily be derived. Indeed, for a control volume CV with surface S , the balance laws for the linear and angular momentum within the geometrically linear theory read

$$\int_S T_i^{(n)} \, dS + \int_{CV} F_i \, d(CV) = \int_{CV} \rho \ddot{u}_i \, d(CV), \quad (14)$$

$$\begin{aligned} & \int_S \left(x_j T_k^{(n)} e_{ijk} + M_i^{(n)} \right) \, dS + \int_{CV} (x_j F_k e_{ijk} + C_i) \, d(CV) \\ & = \int_{CV} \rho x_j \ddot{u}_k e_{ijk} \, d(CV) + \int_{CV} \frac{\rho h^2}{3} \partial_j \ddot{u}_k e_{ijk} \, d(CV), \end{aligned} \quad (15)$$

where $T_i^{(n)}$ is the surface force per unit area (force traction), F_i is the body force per unit volume, $M_i^{(n)}$ is the surface moment per unit area (couple traction), C_i is the body moment per unit volume, (\cdot) denotes now $\partial(\cdot)/\partial t$, and x_j are the components of the position vector of each material particle with elementary volume $d(CV)$. The second term in the RHS of (15) represents the effect of micro-inertia.

2.2. Stresses, equation of motion and constitutive relations

Next, pertinent *force-stress* and *couple-stress* tensors are introduced by considering the equilibrium of the elementary material tetrahedron and enforcing (14) and (15), respectively. The force-stress or total stress tensor σ_{ij} (which is asymmetric) is defined by

$$T_i^{(n)} = \sigma_{ji}n_j, \quad (16)$$

and the couple-stress tensor μ_{ij} (which also is asymmetric) by

$$M_i^{(n)} = \mu_{ji}n_j, \quad (17)$$

where n_j are the direction cosines of the outward unit vector \mathbf{n} , which is normal to the surface. In addition, just like the third Newton's law $\mathbf{T}^{(n)} = -\mathbf{T}^{(-n)}$ is proved to hold by considering the equilibrium of a material 'slice', it can also be proved that $\mathbf{M}^{(n)} = -\mathbf{M}^{(-n)}$. The couple-stresses μ_{ij} are expressed in dimensions of [force][length]⁻¹. Further, σ_{ij} can be decomposed into a symmetric and an anti-symmetric part

$$\sigma_{ij} = \tau_{ij} + \alpha_{ij}, \quad (18)$$

with $\tau_{ij} = \tau_{ji}$ and $\alpha_{ij} = -\alpha_{ji}$, whereas it is advantageous—as will become clear below—to decompose μ_{ij} into its deviatoric $\mu_{ij}^{(D)}$ and spherical $\mu_{ij}^{(S)}$ part in the following manner

$$\mu_{ij} = m_{ij} + \frac{1}{3}\delta_{ij}\mu_{kk}, \quad (19)$$

where $m_{ij} \equiv \mu_{ij}^{(D)}$, $\mu_{ij}^{(S)} \equiv (1/3)\delta_{ij}\mu_{kk}$, and δ_{ij} is the Kronecker delta.

Now, with the above definitions in hand and with the help of the divergence theorem, one may obtain the stress equations of motion. Thus, Eq. (15) leads to the following moment equation

$$\partial_i \mu_{ij} + \sigma_{ki}e_{ijk} + C_j = \frac{\rho h^2}{3} \partial_k \ddot{u}_i e_{ijk}, \quad (20)$$

which can also be written as

$$\frac{1}{2} \partial_i \mu_{il} e_{jkl} + \alpha_{jk} + \frac{1}{2} C_l e_{jkl} = \frac{\rho h^2}{3} \partial_m \ddot{u}_n e_{nim} e_{jik}, \quad (21)$$

since by its definition the anti-symmetric part of stress is written as $\alpha \equiv -(1/2)\mathbf{I} \times (\boldsymbol{\sigma} \times \mathbf{I})$, where \mathbf{I} is the idemfactor (i.e. the dyadic representation of the Kronecker delta). Also, Eq. (14) leads to the following force equation

$$\partial_j \sigma_{jk} + F_k = \rho \ddot{u}_k, \quad (22)$$

or, by virtue of (18), to the equation

$$\partial_j \tau_{jk} + \partial_j \alpha_{jk} + F_k = \rho \ddot{u}_k. \quad (23)$$

Further, combining (21) and (23) yields the *single* equation

$$\partial_j \tau_{jk} - \frac{1}{2} \partial_j \partial_i \mu_{il} e_{jkl} + F_k - \frac{1}{2} \partial_j C_l e_{jkl} = \rho \ddot{u}_k - \frac{\rho h^2}{3} \partial_{jj} \ddot{u}_k. \quad (24)$$

Finally, in view of Eq. (19) and by taking into account that $\text{curl}(\text{div}((1/3)\delta_{ij}\mu_{kk})) = 0$, i.e. that the curl of the divergence of a spherical tensor vanishes, we write (24) as

$$\partial_j \tau_{jk} - \frac{1}{2} \partial_j \partial_i m_{il} e_{jkl} + F_k - \frac{1}{2} \partial_j C_l e_{jkl} = \rho \ddot{u}_k - \frac{\rho h^2}{3} \partial_{jj} \ddot{u}_k. \quad (25)$$

Eq. (25) is therefore the single *equation of motion*. It should also be noticed that, within the present theory and in the general inertial case, the total stress tensor σ_{ij} is not an objective quantity since Eqs. (18) and (21) imply that σ_{ij} contains acceleration terms, which are non-objective of course. Body forces and body couples are also non-objective quantities. On the other hand, σ_{jk} and μ_{jk} should be objective because these are related by (16) and (17) with the surface loads, which are objective quantities. The foregoing difficulty—identified by Eringen (1968) among others—can be circumvented by the following suggestion of Jaunzemis (1967, p. 233). To render σ_{jk} objective, α_{jk} in (21) is taken objective by assuming that there exists an *effective* body couple defined as the difference $[(\rho h^2/3)\partial_m \ddot{u}_n e_{nim} e_{jik} - (1/2)C_l e_{jkl}]$ that is objective, although none of the two quantities in the brackets is objective. Of course, one may be opposed to the previous argument but cannot object to the fact that in the absence of body couples and in particular (but still useful) cases of the theory, like the quasi-static case and the time-harmonic steady-state dynamic case, σ_{ij} becomes objective. Indeed, in the quasi-static case the acceleration terms are zero, whereas in the time-harmonic case (which is used below to analyze the problem of Rayleigh-wave motions) the accelerations \ddot{u}_k in the plane-strain case (now $k = (1, 2)$) become $-\omega^2 u_k(x_1, x_2)$ through Eq. (35a,b)—see next section. It is noticed that ω is the frequency and the time is factored out through the common term $\exp(-i\omega t)$ multiplying every quantity in a time-harmonic response. Therefore, non-objective quantities do not appear in the case of interest here.

In light of the above discussion, one may view the general transient theory as a vehicle to arrive at specific cases of interest, without omitting micro-structure effects, even if the general theory exhibits the foregoing peculiarity. On the other hand, the constitutive equations obey the principle of objectivity in any case (cf. Eqs. (31)–(33) below).

As for the kinematical description of the continuum, the following quantities are defined

$$\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j), \quad (26)$$

$$\omega_{ij} = \frac{1}{2}(\partial_i u_j - \partial_j u_i), \quad (27)$$

$$\omega_i = \frac{1}{2}e_{ijk}\partial_j u_k, \quad (28)$$

$$\kappa_{ij} = \partial_i \omega_j, \quad (29)$$

where ε_{ij} is the strain tensor, ω_{ij} is the rotation tensor, ω_i is the rotation vector, and κ_{ij} is the torsion-flexure tensor (i.e. the gradient of the rotation or the curl of the strain) expressed in dimensions of $[\text{length}]^{-1}$. Notice also that (29) can alternatively be written as

$$\kappa_{ij} = \frac{1}{2}e_{jkl}\partial_l \partial_k u_i = e_{jkl}\partial_k \varepsilon_{li}. \quad (30)$$

We notice, in addition, that $\kappa_{ii} = 0$ because $\kappa_{ii} \equiv \partial_i \omega_i = (1/2)e_{ijk}u_{k,ji} = 0$ (where the latter equality is true due to the skew-symmetry of the permutation symbol) and, therefore, that κ_{ij} has only eight independent components. The tensor κ_{ij} is obviously an *asymmetric* tensor.

It is time now to introduce the constitutive equations of the theory by assuming the following *isotropic* expression for the potential-energy density \mathcal{W} . This expression involves four different material constants and it reads

$$\mathcal{W} \equiv \mathcal{W}(\varepsilon_{ij}, \kappa_{ij}) = \frac{1}{2}\lambda \varepsilon_{ii} \varepsilon_{jj} + \mu \varepsilon_{ij} \varepsilon_{ij} + 2\eta \kappa_{ij} \kappa_{ij} + 2\eta' \kappa_{ij} \kappa_{ji}, \quad (31)$$

where $(\lambda, \mu, \eta, \eta')$ are the material constants. Eq. (31) leads to the constitutive equations

$$\tau_{ij} \equiv \sigma_{(ij)} = \frac{\partial W}{\partial \varepsilon_{ij}} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}, \quad (32)$$

$$m_{ij} = \frac{\partial W}{\partial \kappa_{ij}} = 4\eta \kappa_{ij} + 4\eta' \kappa_{ji}. \quad (33)$$

In view of the above, the moduli (λ, μ) have the same meaning with the Lamé constants of the classical elasticity theory, whereas the moduli (η, η') account for the couple-stress effects of the material behavior.

Finally, the following points are of notice. (i) The form in (31) stems from the more general form of the first law of thermodynamics for a continuum with couple-stresses $\rho \dot{E} = \tau_{ij} \dot{\varepsilon}_{ij} + (1/2) m_{ij} e_{jkl} \partial_{kl} \dot{u}_l$, where E is the internal energy per unit mass. (ii) The couple-stress moduli (η, η') are expressed in dimensions of [force]. (iii) Since $\kappa_{ii} = 0$, $m_{ii} = 0$ is also valid and therefore the tensor m_{ij} has only eight independent components. (iv) The scalar $(1/3)\mu_{kk}$ of the couple-stress tensor μ_{ij} does not appear in the final equation of motion and in the constitutive equations either. Consequently, $(1/3)\mu_{kk}$ is left *indeterminate* within the couple-stress theory. In other words, the field μ_{ij} is unique except for an arbitrary additive (constant) isotropic couple-stress field. (v) The following restrictions for the material constants should prevail as obtained by Mindlin and Tiersten (1962) on the basis of a positive definite potential-energy density (the positive definiteness is, in turn, a necessary condition for the uniqueness theorem to be proved—see e.g. Mindlin and Tiersten, 1962 and Mindlin and Eshel, 1968)

$$3\lambda + 2\mu > 0, \quad \mu > 0, \quad \eta > 0, \quad -1 < \frac{\eta'}{\eta} < 1. \quad (34a, b, c, d)$$

3. Plane-strain time-harmonic dynamical response

In order to analyze wave motions, we consider the following two-dimensional time-harmonic response of a linearly elastic isotropic body characterized by micro-structure and couple-stress effects. The body occupies a domain in the $(x \equiv x_1, y \equiv x_2)$ -plane and is under conditions of plane strain. For the analysis of Rayleigh waves, we consider as an appropriate domain the half-space $y \geq 0$ having as a boundary the plane $(x, z \equiv x_3)$. Then, we have

$$u_x(x, y, t) = u_x(x, y) \cdot \exp(-i\omega t), \quad u_y(x, y, t) = u_y(x, y) \cdot \exp(-i\omega t), \quad u_z \equiv 0, \quad (35a, b, c)$$

where (u_x, u_y, u_z) are the displacement components, $i \equiv (-1)^{1/2}$, t is the time, and ω is the frequency. In what follows, as is standard in time-harmonic problems, it is understood that all field quantities are to be multiplied by the factor $\exp(-i\omega t)$ and that the real part of the resulting expression is to be taken. Below, we derive the field equations of the problem and then uncouple them by using Lamé potentials.

First, the components of the force-stress and couple-stress tensors will be obtained. The independence upon the coordinate z of *all* components of force-stress and couple-stress tensors, under the assumption (35c), was proved by Muki and Sternberg (1965). Indeed, it is noteworthy that, contrary to the respective plane-strain case in the conventional theory, this independence is not obvious within the couple-stress theory. Notice further that except for ω_z and $(\kappa_{xz}, \kappa_{yz})$ all other components of the rotation vector and the torsion-flexure tensor identically vanish, in the particular case of plane-strain considered here. The non-vanishing components $(\tau_{xx}, \tau_{xy}, \tau_{yy})$ and (m_{xz}, m_{yz}) follow from (32) and (33), respectively. Then, $(\alpha_{xx}, \alpha_{xy}, \alpha_{yx}, \alpha_{yy})$ are found from (21) and, finally, $(\sigma_{xx}, \sigma_{xy}, \sigma_{yx}, \sigma_{yy})$ are provided by (18). Vanishing body

forces and body couples are assumed in what follows. In view of the above, the following expressions are written

$$m_{xz} = 2\eta \left(\frac{\partial^2 u_y}{\partial x^2} - \frac{\partial^2 u_x}{\partial x \partial y} \right), \quad (36)$$

$$m_{yz} = 2\eta \left(\frac{\partial^2 u_y}{\partial x \partial y} - \frac{\partial^2 u_x}{\partial y^2} \right), \quad (37)$$

$$\alpha_{xx} = \alpha_{yy} = 0, \quad (38)$$

$$\alpha_{yx} = -I\omega^2 \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) + \frac{1}{2} \left(\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} \right), \quad (39)$$

$$\alpha_{xy} = -\alpha_{yx}, \quad (40)$$

$$\sigma_{xx} = (\lambda + 2\mu) \frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_y}{\partial y}, \quad (41)$$

$$\sigma_{yy} = (\lambda + 2\mu) \frac{\partial u_y}{\partial y} + \lambda \frac{\partial u_x}{\partial x}, \quad (42)$$

$$\sigma_{yx} = \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) - I\omega^2 \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) + \eta \left(\frac{\partial^3 u_y}{\partial x^3} - \frac{\partial^3 u_x}{\partial x^2 \partial y} + \frac{\partial^3 u_y}{\partial x \partial y^2} - \frac{\partial^3 u_x}{\partial y^3} \right), \quad (43)$$

$$\sigma_{xy} = \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) + I\omega^2 \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) - \eta \left(\frac{\partial^3 u_y}{\partial x^3} - \frac{\partial^3 u_x}{\partial x^2 \partial y} + \frac{\partial^3 u_y}{\partial x \partial y^2} - \frac{\partial^3 u_x}{\partial y^3} \right), \quad (44)$$

where $I = \rho h^2/3$ is the micro-inertia coefficient and one may notice also that the material constant η' does not appear at all in the plane-strain equations. One may observe that all stresses given above are objective, a fact that is in accord with the relative discussion in the last section.

Next, the field equations of the problem will be obtained. The equation of motion (25) takes the form $\partial_j \tau_{jk} - (1/2) \partial_j \partial_l m_{il} e_{jkl} = \rho \ddot{u}_k - I \partial_{jj} \ddot{u}_k$, where the indices (i, j, k, l) take now the values 1 and 2 only. Further, the second-order time derivatives, in view of (35), will become $-\omega^2 u_k(x, y)$, and the only surviving stresses $(\tau_{xx}, \tau_{xy}, \tau_{yy})$ and (m_{xz}, m_{yz}) will be provided by (32) and (36)–(37), respectively, in terms of the displacement gradients. Therefore, in the case of a time-harmonic plane-strain response, the equation of motion leads to the following system of *coupled* PDEs for the displacement components (u_x, u_y)

$$\begin{aligned} & (\lambda + 2\mu) \frac{\partial^2 u_x}{\partial x^2} + \lambda \frac{\partial^2 u_y}{\partial x \partial y} + \mu \left(\frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_x}{\partial y^2} \right) + \eta \left(\frac{\partial^4 u_y}{\partial x^3 \partial y} - \frac{\partial^4 u_x}{\partial x^2 \partial y^2} + \frac{\partial^4 u_y}{\partial x \partial y^3} - \frac{\partial^4 u_x}{\partial y^4} \right) \\ & = -\rho \omega^2 u_x + I \omega^2 \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right), \end{aligned} \quad (45)$$

$$\begin{aligned} & (\lambda + 2\mu) \frac{\partial^2 u_y}{\partial y^2} + \lambda \frac{\partial^2 u_x}{\partial x \partial y} + \mu \left(\frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_y}{\partial x^2} \right) + \eta \left(\frac{\partial^4 u_x}{\partial x^3 \partial y} - \frac{\partial^4 u_y}{\partial x^2 \partial y^2} + \frac{\partial^4 u_x}{\partial x \partial y^3} - \frac{\partial^4 u_y}{\partial x^4} \right) \\ & = -\rho \omega^2 u_y + I \omega^2 \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right). \end{aligned} \quad (46)$$

Although the above system is much more complicated than that in the respective case of classical elastodynamics (see e.g. Sternberg, 1960; Fung, 1965), uncoupling by the use of Lamé-type potentials still proves to be successful. The potentials (ϕ, ψ) are defined in terms of the displacement components (u_x, u_y) as

$$u_x = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}, \quad u_y = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}, \quad (47a, b)$$

and some tedious algebra leads to the following *uncoupled* PDEs, which are the field equations of the problem. The first PDE is of the second order but the second is of the fourth order, i.e.

$$[(\lambda + 2\mu - I\omega^2)\nabla^2 + \rho\omega^2]\phi = 0, \quad (48)$$

$$[\eta\nabla^4 - (\mu - I\omega^2)\nabla^2 - \rho\omega^2]\psi = 0, \quad (49)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \quad (50a, b)$$

Finally, introducing the following quantities

$$g_L = 1 - \frac{I\omega^2}{\lambda + 2\mu}, \quad g_T = 1 - \frac{I\omega^2}{\mu}, \quad k_L^2 = \frac{\rho\omega^2}{\lambda + 2\mu}, \quad k_T^2 = \frac{\rho\omega^2}{\mu}, \quad (51a, b, c, d)$$

permits writing (48) and (49) under the compact forms

$$g_L \nabla^2 \phi + k_L^2 \phi = 0, \quad (52)$$

$$(\eta/\mu) \nabla^4 \psi - g_T \nabla^2 \psi - k_T^2 \psi = 0, \quad (53)$$

where the coefficient (η/μ) of the operator with the highest order in (53) has dimensions of $[\text{length}]^2$. Since $(\eta/\mu) \rightarrow 0$ in the case of classical elasticity, the very form of (53) reveals the *singular-perturbation* character of the couple-stress theory and the emergence of associated *boundary-layer* effects. It is expected therefore that the influence of couple-stresses hinges crucially on the relative size of the characteristic length-parameter $(\eta/\mu)^{1/2}$. It is reminded that one of the goals of the present work is to provide means for estimating the relation between $(\eta/\mu)^{1/2}$ and the size of the material unit-cell $2h$. It is also noticed that in the case $g_L = 1$ (i.e. when the micro-inertia is absent), (52) becomes identical to the Helmholtz PDE governing the longitudinal time-harmonic motions within classical elastodynamics. On the other hand, unlike the corresponding case of shear motions within classical elastodynamics, Eq. (53) being it a fourth-order PDE shows that wave signals emitted from a disturbance point propagate at different velocities (see e.g. Chester, 1971).

4. General transformed solutions and the dispersion equation for Rayleigh waves

The two-sided Laplace transform is utilized to suppress the x -dependence in the field equations and boundary conditions in the half-space domain $(-\infty < x < \infty, y \geq 0)$ and to lead to general solutions for the potentials in the complex domain. The direct and inverse transforms are defined as follows (see e.g. Bracewell, 1965)

$$f^*(p, y) = \int_{-\infty}^{\infty} f(x, y) e^{-px} dx, \quad f(x, y) = \frac{1}{2\pi i} \int_{\text{Br}} f^*(p, y) e^{px} dp, \quad (54a, b)$$

where Br denotes the Bromwich inversion path within the region of analyticity of the function $f^*(p, y)$. Transforming Eqs. (52) and (53) with (54a) gives the following ODEs

$$g_L \frac{d^2 \phi^*}{dy^2} + (g_L p^2 + k_L^2) \phi^* = 0, \quad (55)$$

$$\frac{\eta}{\mu} \frac{d^4 \psi^*}{dy^4} + \left(2 \frac{\eta}{\mu} p^2 - g_T \right) \frac{d^2 \psi^*}{dy^2} + \left(\frac{\eta}{\mu} p^4 - g_T p^2 - k_T^2 \right) \psi^* = 0, \quad (56)$$

and, further, the general *transformed* solutions for $y \geq 0$

$$\phi^*(p, y) = A_1(p) \cdot \exp(-\beta_L y), \quad (57)$$

$$\psi^*(p, y) = A_2(p) \cdot \exp(-\beta_T y) + A_3(p) \cdot \exp(-\gamma_T y), \quad (58)$$

where

$$\beta_L = i(p^2 + \sigma_L^2)^{1/2}, \quad \text{with} \quad \sigma_L = \frac{k_L}{g_L^{1/2}}, \quad (59a, b)$$

$$\beta_T = i(p^2 + \sigma_T^2)^{1/2}, \quad \text{with} \quad \sigma_T = \frac{[(g_T^2 + 4(\eta/\mu)k_T^2)^{1/2} - g_T]^{1/2}}{(2\eta/\mu)^{1/2}}, \quad (60a, b)$$

$$\gamma_T = (\tau_T^2 - p^2)^{1/2}, \quad \text{with} \quad \tau_T = \frac{[(g_T^2 + 4(\eta/\mu)k_T^2)^{1/2} + g_T]^{1/2}}{(2\eta/\mu)^{1/2}}, \quad (61a, b)$$

and the functions (A_1, A_2, A_3) are yet unknown functions (*amplitude* functions), which can be determined in each specific problem through the enforcement of boundary conditions.

A point also that deserves attention is the introduction of the branch cuts for the functions $(\beta_L, \beta_T, \gamma_T)$ in the complex p -plane, in such a manner that a bounded solution at $y \rightarrow \infty$ is always secured (i.e. in order for the functions to have positive real parts). Any inversion then of the type (54b) should be performed by considering the appropriate restrictions in the cut plane. Fig. 3 shows the branch cuts of the functions (β_T, γ_T) for *all* frequencies and the branch cuts of the function β_L for those frequencies resulting only in *real* σ_L (the latter condition requires that $g_L > 0$). In the case that $g_L < 0$, β_L in (59a) takes the form $\beta_L = (\sigma_L^2 - p^2)^{1/2}$, where $\sigma_L = k_L/(|g_L|)^{1/2}$, and the branch cuts for β_L will be along the intervals $(\sigma_L < |\text{Re}(p)| < \infty, \text{Im}(p) = 0)$ (i.e. the cuts will resemble the ones for γ_T). At the frequency yielding $g_L = 0$, we have a change of the *character* of the PDE in (52) from a standard Helmholtz equation (when $g_L > 0$) to a modified Helmholtz equation (when $g_L < 0$). However, when material constants for usual solids are employed, the latter case occurs at extremely high frequencies. Finally, Fig. 3 also exhibits the behavior of the functions $(\beta_L, \beta_T, \gamma_T)$ in the complex p -plane.

The criterion now for surface Rayleigh waves is that the displacement potentials decay exponentially with the distance y from the half-space surface (Knowles, 1966; see also Brock, 1998 for a general discussion of surface waves within the classical elasticity theory). In view of the previous analysis leading to (57) and (58), we explore the possibility of *progressive-wave* solutions to (25) having the following form of a distinct harmonic component

$$\begin{aligned} \bar{\phi}_s(x, y, t) &= A_1(p) \cdot \exp(-|\beta_L|y) \cdot \exp(px) \cdot \exp[-i\omega(p) \cdot t] \equiv \phi_s^*(p, y) \cdot \exp(px) \cdot \exp[-i\omega(p) \cdot t] \\ &\equiv \phi_s^*(q, y) \cdot \exp[iq(x - C_{ph}t)], \end{aligned} \quad (62)$$

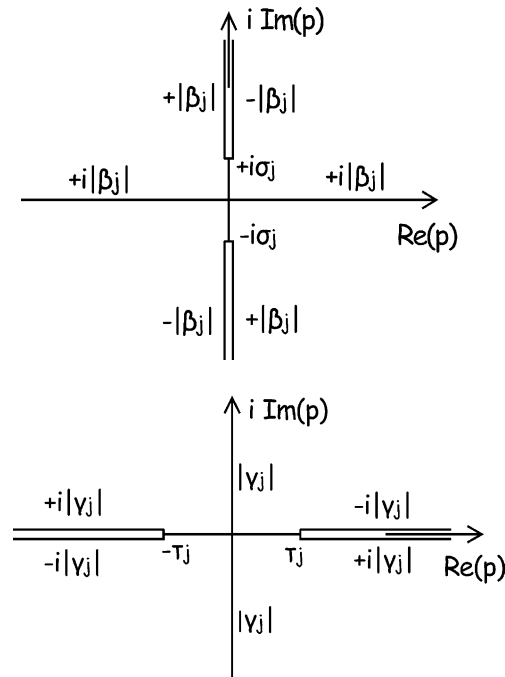


Fig. 3. Branch cuts for the functions $(\beta_L, \beta_T, \gamma_T)$. The index j takes the values (L, T) .

$$\begin{aligned}\bar{\psi}_s(x, y, t) &= \{A_2(p) \cdot \exp(-|\beta_T|y) + A_3(p) \cdot \exp(-|\gamma_T|y)\} \cdot \exp(px) \cdot \exp[-i\omega(p) \cdot t] \\ &\equiv \psi_s^*(p, y) \cdot \exp(px) \cdot \exp[-i\omega(p) \cdot t] \equiv \psi_s^*(q, y) \cdot \exp[iq(x - C_{ph}t)],\end{aligned}\quad (63)$$

where (A_1, A_2, A_3) represent arbitrary amplitude functions denoting the relative dominance of a particular harmonic component, the propagation wavenumber $q \equiv (p/i)$ is taken to be a real quantity, $C_{ph} = \omega/q$ is defined as the phase velocity of the Rayleigh waves, and $(\beta_L, \beta_T, \gamma_T)$ defined in (59a), (60a) and (61a) are taken to be *real* and *positive* functions. The latter restriction is satisfied if and only if $\sigma_L < |q|$ and $\sigma_T < |q|$. Taking a *real* wavenumber excludes the possibility of localized *standing* waves (i.e. leaky or evanescent motions), whereas the frequency at which the wavenumber (for a particular mode) changes from real to imaginary (or complex) values is the cut-off frequency. Finally, we notice that a general Rayleigh surface-wave solution (synthesis) can be derived from (62) and (63) as the following Laplace inversion integrals

$$\phi(x, y, t) = \frac{1}{2\pi i} \int_{Br} \phi_s^*(p, y) e^{px} e^{-i\omega(p) \cdot t} dp, \quad \psi(x, y, t) = \frac{1}{2\pi i} \int_{Br} \psi_s^*(p, y) e^{px} e^{-i\omega(p) \cdot t} dp.$$

Next, the appropriate *traction-free* boundary conditions are considered for unforced Rayleigh-wave propagation. These follow from Eqs. (16) and (17) as

$$\sigma_{yy}(x, y=0) = 0, \quad \sigma_{yx}(x, y=0) = 0, \quad m_{yz}(x, y=0) = 0. \quad (64a, b, c)$$

In view now of the constitutive relations in (37), (42) and (43), the definition of the Lamé potentials in (47) and the properties of the two-sided Laplace transform, the above equations provide

$$(\lambda + 2\mu) \frac{d^2 \phi^*(p, y=0)}{dy^2} - 2\mu p \frac{d\psi^*(p, y=0)}{dy} + \lambda p^2 \phi^*(p, y=0) = 0, \quad (65a)$$

$$(\mu - I\omega^2) \frac{d^2\psi^*(p, y=0)}{dy^2} + 2\mu p \frac{d\phi^*(p, y=0)}{dy} - (\mu - I\omega^2)p^2\psi^*(p, y=0) - \eta \left(\frac{d^4\psi^*(p, y=0)}{dy^4} + p^4\psi^*(p, y=0) + 2p^2 \frac{d^2\psi^*(p, y=0)}{dy^2} \right) = 0, \quad (65b)$$

$$\frac{d^3\psi^*(p, y=0)}{dy^3} + p^2 \frac{d\psi^*(p, y=0)}{dy} = 0. \quad (65c)$$

Further, introducing the general solutions (57) and (58) in Eq. (65), one obtains the following system of three equations in the three unknowns (A_1, A_2, A_3)

$$\{\lambda p^2 + (\lambda + 2\mu)\beta_L^2\}A_1(p) + 2\mu p\beta_T A_2(p) + 2\mu p\gamma_T A_3(p) = 0, \quad (66a)$$

$$-2p\beta_L A_1(p) + \{-(\eta/\mu)(\beta_T^4 + 2\beta_T^2 p^2 + p^4) - p^2 g_T + \beta_T^2 + \beta_T^2(I\omega^2/\mu)\}A_2(p) + \{-(\eta/\mu)(\gamma_T^4 + 2\gamma_T^2 p^2 + p^4) - p^2 g_T + \gamma_T^2 + \gamma_T^2(I\omega^2/\mu)\}A_3(p) = 0, \quad (66b)$$

$$\beta_T(p^2 + \beta_T^2)A_2(p) + \gamma_T(p^2 + \gamma_T^2)A_3(p) = 0. \quad (66c)$$

With the understanding that the zero RHS in each of the above equations could be replaced by appropriate transformed expressions, Eqs. (66) may serve for the solution to general traction boundary-value-problems involving *forced* motions and not only Rayleigh-type free motions.

If, additionally, the wavenumber q is introduced (as defined above) and the expressions $|\beta_L| = (q^2 - \sigma_L^2)^{1/2}$ ($g_L > 0$ case), $|\beta_T| = (q^2 - \sigma_T^2)^{1/2}$ and $|\gamma_T| = (\tau_T^2 + q^2)^{1/2}$ (with q being real and such that $\sigma_L < |q|$ and $\sigma_T < |q|$) are used in (66), the following system is obtained

$$[-\lambda\sigma_L^2 + 2\mu(q^2 - \sigma_L^2)]A_1(q) + [2\mu q(q^2 - \sigma_T^2)^{1/2}]iA_2(q) + [2\mu q(q^2 + \tau_T^2)^{1/2}]iA_3(q) = 0, \quad (67a)$$

$$[-2q(q^2 - \sigma_L^2)^{1/2}]iA_1(q) + \left[-(\eta/\mu)\sigma_T^4 + q^2 g_T + (q^2 - \sigma_T^2) \left(1 + \frac{I\omega^2}{\mu} \right) \right] A_2(q) + \left[-(\eta/\mu)\tau_T^4 + q^2 g_T + (q^2 + \tau_T^2) \left(1 + \frac{I\omega^2}{\mu} \right) \right] A_3(q) = 0, \quad (67b)$$

$$[-\sigma_T^2(q^2 - \sigma_T^2)^{1/2}]A_2(q) + [\tau_T^2(q^2 + \tau_T^2)^{1/2}]A_3(q) = 0. \quad (67c)$$

In the case $g_L < 0$, the expression $(q^2 - \sigma_L^2)^{1/2}$ for $|\beta_L|$ will be replaced by $(q^2 + \sigma_L^2)^{1/2}$ in Eqs. (66).

The vanishing now of the determinant of the coefficients of the unknowns (eigenvalue problem) provides the *dispersion* equation for the propagation of Rayleigh waves in a micro-structured material characterized by couple-stress elasticity

$$[-2q(q^2 - \sigma_L^2)^{1/2}][-\sigma_T^2(q^2 - \sigma_T^2)^{1/2}][2\mu q(q^2 + \tau_T^2)^{1/2}] - [-2q(q^2 - \sigma_L^2)^{1/2}][\tau_T^2(q^2 + \tau_T^2)^{1/2}][2\mu q(q^2 - \sigma_T^2)^{1/2}] - [-\lambda\sigma_L^2 + 2\mu(q^2 - \sigma_L^2)][\tau_T^2(q^2 + \tau_T^2)^{1/2}] \left[-(\eta/\mu)\sigma_T^4 + q^2 g_T + (q^2 - \sigma_T^2) \left(1 + \frac{I\omega^2}{\mu} \right) \right] + [-\lambda\sigma_L^2 + 2\mu(q^2 - \sigma_L^2)][-\sigma_T^2(q^2 - \sigma_T^2)^{1/2}] \left[-(\eta/\mu)\tau_T^4 + q^2 g_T + (q^2 + \tau_T^2) \left(1 + \frac{I\omega^2}{\mu} \right) \right] = 0. \quad (68)$$

The above equation is complicated and does not yield even to modern programs of symbolic algebra (e.g. MATHEMATICA™). Numerical results were therefore derived through solving the dispersion equation numerically by techniques of root bracketing and root finding with bisection (Brent's method—see e.g. Press et al., 1986). This requires a rather elaborate FORTRAN programming though.

5. Numerical results and concluding remarks

Here, we give some representative results in the form of dispersion curves. To this end, the following normalizations are employed for, respectively, the dimensionless wavenumber and the dimensionless frequency

$$q_d = (\eta/\mu)^{1/2} q, \quad \omega_d = \frac{h}{3^{1/2}(\mu/\rho)^{1/2}} \omega, \quad (69a, b)$$

whereas the shear-wave velocity of the medium in the *absence* of couple-stress and micro-structure effects $V_T = (\mu/\rho)^{1/2}$ is utilized to conveniently normalize the phase velocity of the Rayleigh waves C_{ph} .

The dispersion curves depicted in the graphs of Figs. 4–9 were obtained by using the properties of a closed-cell polymethacrylimide material exhibiting micro-structure. These properties are given by Anderson

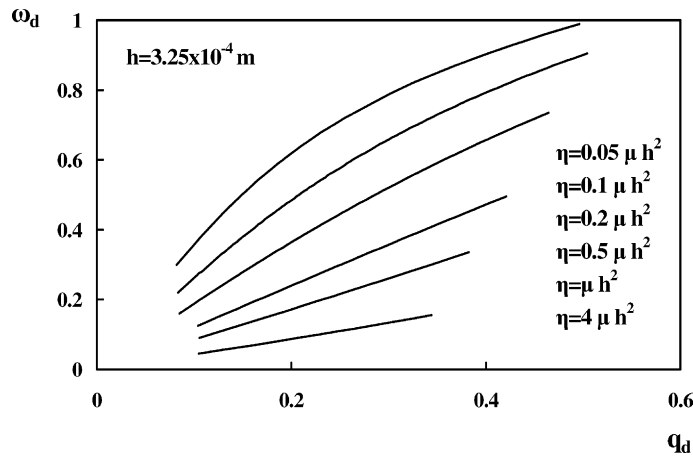


Fig. 4. Dispersion curves for Rayleigh waves showing the variation of the normalized frequency ω_d with the normalized wavenumber q_d for a closed-cell polymethacrylimide material exhibiting micro-structure. Graphs for six different relations between the couple-stress modulus η and the characteristic material length h are presented.

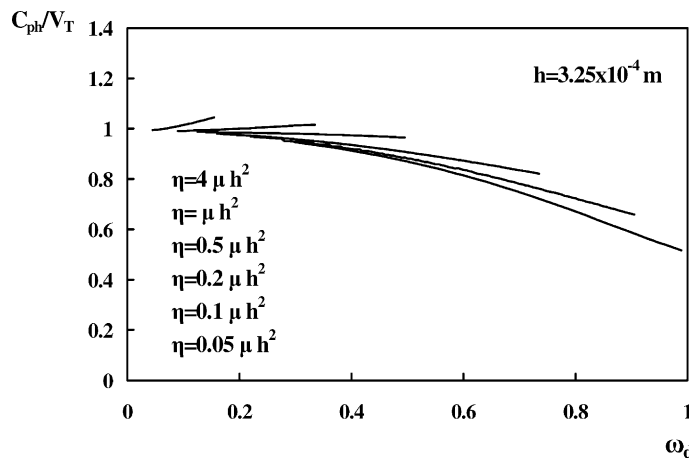


Fig. 5. Dispersion curves for Rayleigh waves showing the variation of the normalized phase velocity C_{ph}/V_T with the normalized frequency ω_d for a closed-cell polymethacrylimide material exhibiting micro-structure. Graphs for six different relations between the couple-stress modulus η and the characteristic material length h are presented.

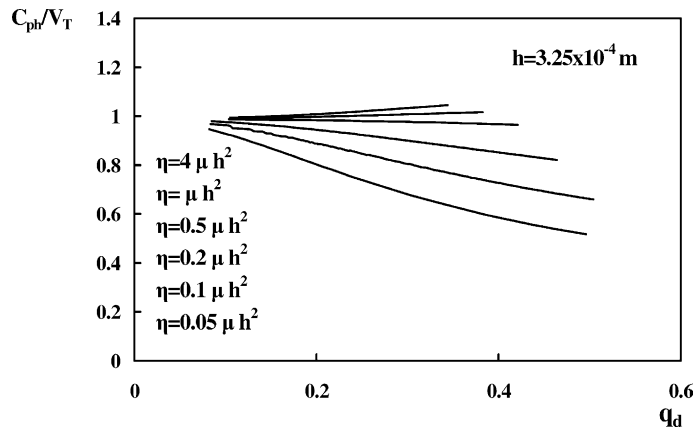


Fig. 6. Dispersion curves for Rayleigh waves showing the variation of the normalized phase velocity C_{ph}/V_T with the normalized wavenumber q_d for a closed-cell polymethacrylimide material exhibiting micro-structure. Graphs for six different relations between the couple-stress modulus η and the characteristic material length h are presented.

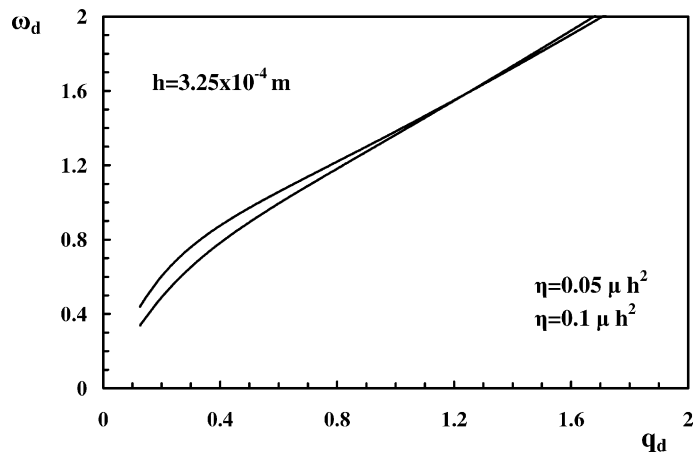


Fig. 7. ω_d vs. q_d dispersion curves for the relations $\eta = 0.05\mu h^2$ and $\eta = 0.1\mu h^2$.

and Lakes (1994) as $\lambda = 99.03 \times 10^6$ Pa, $\mu = 285 \times 10^6$ Pa, $\nu = 0.13$ (Poisson's ratio), $\rho = 0.38$ gr/cm³ and $h = 3.25 \times 10^{-4}$ m. In order to compare the form of the dispersion curves obtained by the present analysis with the form of the dispersion curves provided by atomic-lattice calculations, we consider several relations between the couple-stress modulus η and the intrinsic material length h . These relations vary from $\eta = 4\mu h^2$ to $\eta = 0.05\mu h^2$. Also, we focus attention only to results derived for high frequencies.

Figs. 4 and 7 clearly show that the graph of ω_d as a function of q_d is no longer a straight line in most of the cases and, therefore, the Rayleigh waves are dispersive. The same conclusion is reached, of course, by observing Figs. 5, 6, 8 and 9, which exhibit the fact that the phase velocity depends upon the frequency and the wavenumber. On the other hand, it may be concluded from the results of Figs. 5 and 6 that the choices $\eta = 4\mu h^2$ and $\eta = \mu h^2$ are rather irrelevant since they lead to Rayleigh-wave propagation speeds higher than or equal to the speed $V_T = (\mu/\rho)^{1/2}$. This phenomenon was never detected in experiments. Between the remaining cases now, the relations $\eta = 0.1\mu h^2$ and $\eta = 0.05\mu h^2$ seem to be appropriate since the dispersion curves they generate (see Figs. 4, 7–9) are close in form to available atomic-lattice calculations (see e.g.

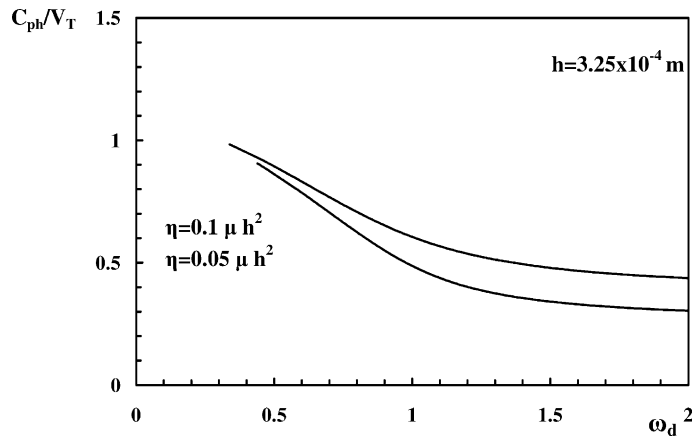


Fig. 8. C_{ph}/V_T vs. ω_d dispersion curves for the relations $\eta = 0.05\mu h^2$ and $\eta = 0.1\mu h^2$.

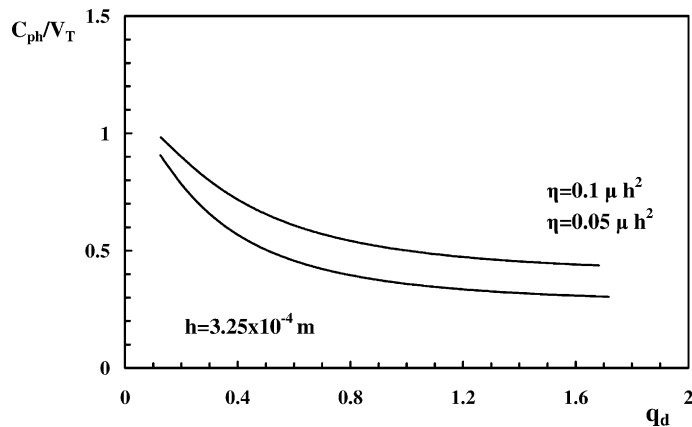


Fig. 9. C_{ph}/V_T vs. q_d dispersion curves for the relations $\eta = 0.05\mu h^2$ and $\eta = 0.1\mu h^2$.

Gazis et al., 1960). Certainly, availability of experimental measurements for a specific material with micro-structure will allow for more accurate estimates of the value of the couple-stress modulus η . We emphasize that the purpose of the comparisons above was just to indicate a means to determine the micro-structural parameter η .

Finally, Fig. 10 shows two dispersion curves derived for a material with constants (λ, μ, ρ) as given before and $\eta = 1.5051$ N. For the purpose of comparison again, in the first case $h = 3.25 \times 10^{-4}$ m was taken, whereas in the second $h = 0$ was set. The latter case concerns a material without micro-structure and the corresponding dispersion curve exhibits an initial part, which is concave up. This finding agrees with the results of Ottosen et al. (2000) showing that the Rayleigh-wave velocity will exceed the shear-wave velocity. As mentioned before, this result is not satisfactory. In addition, the form of the dispersion curve for $h = 0$ does not agree with that obtained by atomic-lattice calculations (see e.g. Gazis et al., 1960). Therefore, the present results demonstrate that the material micro-structure is a *necessity* for a satisfactory simulation of dispersion curves obtained by the atomic-lattice approach.

In conclusion, the main goals of the present work were: (i) to derive in a simple manner basic theoretical results concerning combined couple-stress and micro-structure effects, (ii) to prove that the couple-stress

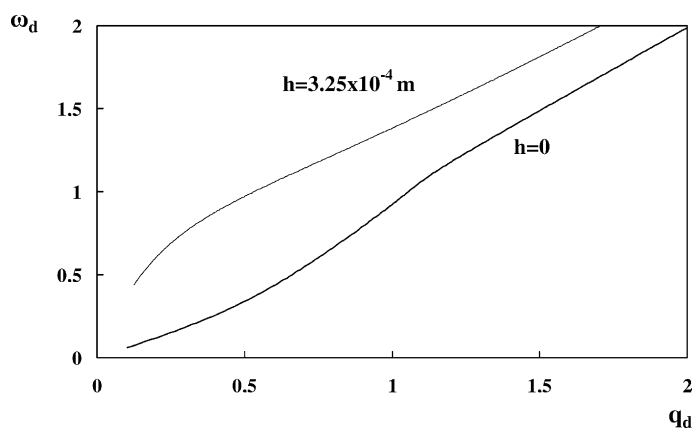


Fig. 10. Two dispersion curves of the type ω_d vs. q_d for the cases $h = 3.25 \times 10^{-4} \text{ m}$ (material with micro-structure) and $h = 0$ (material without micro-structure). The couple-stress modulus is $\eta = 1.5051 \text{ N}$ and the constants (λ, μ, ρ) are those of the closed-cell polymethacrylimide material.

elasticity theory with micro-structure does predict *dispersive* Rayleigh waves at high frequencies, and (iii) to indicate that comparing the form of dispersion curves obtained by the present approach with the form obtained experimentally or by the discrete particle theory may determine the relation between the couple-stress modulus η and the size of the unit cell h of the micro-structured material. Our study shows that the use of the couple-stress elasticity theory with micro-structure in the problem of Rayleigh waves extends the range of applicability of continuum theories. This is a step in the efforts towards bridging the gap between classical (monopolar) theories of continua and theories of atomic lattices.

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